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Exact solution of close-packed dimers on the kagomé lattice

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It is well known that exact enumerations of close-packed dimers can be carried out for two-dimensional lattices. While details of results are now known for most lattices, due to the unique nature of the lattice structure, there has been no complete analysis for the kagomé lattice. Here we derive the close-form expression $(1/3)\ \ln(4xyz)$ for the free energy of close-packed dimers on the kagomé lattice, where x, y, and z are dimer weights. We use two different approaches: the Kasteleyn method of evaluating a Pfaffian and an alternative vertex model formulation. Both methods lead to the same final expression. The correlation function between two dimers at a distance equal or greater than two lattice spacings is found to vanish identically.

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I. INTRODUCTION

A central problem in statistical physics is the enumeration of close-packed dimers on lattices. The origin of the problem has a long history tracing back to the 1937 paper of Fowler and Rushbrooke [1] in their attempt at enumerating the absorption of diatomic molecules on a surface. A milestone in the history of the dimer problem is the exact solution for the square lattice obtained by Kasteleyn [2] and Temperley and Fisher [3] in 1961. Indeed, the method of Kasteleyn is quite general and applicable to all planar lattices [4]. Exact results obtained in this way are summarized in a recent review [5] for a number of two-dimensional lattices.

In the case of the kagomé lattice, however, there has been no complete analysis of the dimer problem other than studies of pure dimer enumerations, most of which are numerical and series expansions (see [6] and references cited therein). In recent years there has been considerable interest in the study of physical phenomena on the kagomé lattice. These range from high-T_c superconductivity [7], Heisenberg antiferromagnets [8–12], and quantum dimers [13] to the occurrence of spin-liquid states [14]. It has also been shown that the consideration of close-packed dimers is related to the ground state of a quantum dimer model [15]. In light of these developments, it is of pertinent interest to take a fresh look at close-packed dimers for the kagomé lattice.

In this Rapid Communication we consider this problem and derive the closed-form expression

$$f_{\text{kagomé}}(x,y,z) = (1/3)\ln(4xyz) \tag{1}$$

for the free energy (for definition of terms see below), a formula quoted in [5]. As exact solutions for other lattices are invariably of the form of a double integral akin to the Onsager solution of the Ising model [5], the very simple expression of the solution (1) and its logarithmic dependence on dimer weights are novel and unique. It points to the special role played by the kagomé lattice (which often makes a problem more amenable) and suggests that caution must be taken in generalizing physical results derived from the kagomé lattice. For example, as we shall see in Sec. IV below, the correlation between two dimers is identically zero at distances equal or greater than two lattice spacings on the kagomé lattice, but this conclusion does not hold for other lattices.

The kagomé lattice is shown in Fig. 1, where x, y, and z are dimer weights along the three principal directions. We denote the lattice by $\mathcal{L}$. Let N (=even) be the number of sites of $\mathcal{L}$, so the lattice can be completely covered by $N/2$ dimers. The dimer generating function is defined to be the summation

$$Z_{\text{kagomé}}(x,y,z) = \sum_{\text{dimer coverings}} x^{n_x} y^{n_y} z^{n_z} \tag{2}$$

over all close-packed dimer configurations of $\mathcal{L}$. Here, n_x is the number of dimers with weight x, etc., subject to $n_x+n_y+n_z=N/2$. Our goal is to evaluate the *per-dimer* free energy

$$f_{\text{kagomé}}(x,y,z) = \lim_{N\to\infty} \frac{1}{N/2} \ln Z_{\text{kagomé}}(x,y,z) \tag{3}$$

in a close form. Past attempts have been confined to enumerations of $f(1,1,1)$. Here we consider the problem for general x, y, and z, a consideration which can find application to anisotropic kagomé systems such as the volborthite antiferromagnet [12].

We derive the solution (1) using two different methods: the Kasteleyn method of evaluating a Pfaffian and alternately a method of a vertex model formulation, which we describe in the next two sections.

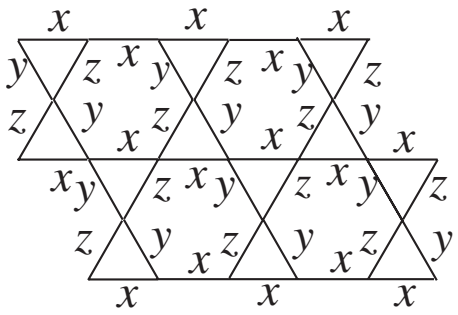

FIG. 1. The kagomé lattice $\mathcal{L}$ with dimer weights x, y, and z in the three principal directions.

II. PFAFFIAN APPROACH

The first step of the Pfaffian method is to find a Kasteleyn orientation [2] of lattice edges. A Kasteleyn orientation of a planar lattice is an orientation of edges such that every transition cycle consisting of a loop of edges derived from the superposition of two dimer coverings has an odd number of arrows pointing in the clockwise direction, a property which we term as *clockwise odd*. While Kasteleyn [4] has demonstrated that such an orientation is possible for all planar graphs, the actual orientation of edges for a given lattice, or graph, still needs to be worked out and the crux of the matter of the Kasteleyn method is the finding of the appropriate clockwise-odd orientation.

For the kagomé lattice a Kasteleyn orientation can be taken as that shown in Fig. 2. The kagomé lattice is composed of up-pointing and down-pointing triangles. The orientation in Fig. 2 consists of orienting all up-pointing triangles and every other down-pointing triangles in the counterclockwise direction, with the other half of the down-pointing triangles oriented as shown. In this orientation a unit cell of the lattice consists of the six sites forming two neighboring down-pointing triangles, which are numbered 1,…,6 as shown. Our orientation is different from that used in [6].

To see that the orientation in Fig. 2 is indeed a Kasteleyn orientation, we note that all transition cycles in Fig. 2 are clockwise odd. As all transition cycles on the lattice can be formed by deforming those in Fig. 2 without altering the clockwise odd property, all transition cycles are also clockwise odd so the orientation in Fig. 2 is a good Kasteleyn orientation [16]. This is essentially the argument of Kasteleyn [4].

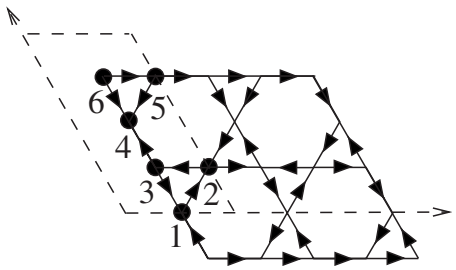

FIG. 2. The Kasteleyn orientation of the kagomé lattice. A unit cell is the region bounded by dashed lines containing the six sites numbered 1,…,6.

After rendering the Kasteleyn orientation, we can write the dimer partition function as a Pfaffian. As the analysis is now standard we follow the standard procedure [see, for example, Eq. (4.34) of [17]] and arrive at the result

$$f_{\text{kagomé}}(x,y,z) = \left(\frac{1}{3}\right)\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \ln \mathrm{Pf}[M(\theta,\phi)]d\theta\, d\phi$$
$$= \left(\frac{1}{6}\right)\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \ln \det|M(\theta,\phi)|d\theta\, d\phi. \tag{4}$$

Here the factor of $1/3$ in the first line comes from the fact that there are three dimers per unit cell in a close-packed configuration, and $\mathrm{Pf}[M(\theta,\phi)] = \sqrt{\det|M(\theta,\phi)|}$ is the Pfaffian of the 6×6 matrix

$$M(\theta,\phi) = a(0,0) + a(1,0)e^{i\theta} + a(-1,0)e^{-i\theta} + a(0,1)e^{i\phi} + a(0,-1)e^{-i\phi} + a(1,1)e^{i(\theta+\phi)} + a(-1,-1)e^{-i(\theta+\phi)}$$
$$= \begin{pmatrix} 0 & z & -y & 0 & ze^{-i(\theta+\phi)} & -ye^{-i\phi} \\ -z & 0 & x(1+e^{i\theta}) & -ze^{i\theta} & 0 & 0 \\ y & -x(1+e^{-i\theta}) & 0 & y & 0 & 0 \\ 0 & ze^{-i\theta} & -y & 0 & -z & -y \\ -ze^{i(\theta+\phi)} & 0 & 0 & z & 0 & x(-1+e^{i\theta}) \\ ye^{i\phi} & 0 & 0 & y & x(1-e^{-i\theta}) & 0 \end{pmatrix}. \tag{5}$$

The a matrices are read off from Fig. 2 to be

$$a(0,0) = \begin{pmatrix} 0 & z & -y & 0 & 0 & 0 \\ -z & 0 & x & 0 & 0 & 0 \\ y & -x & 0 & y & 0 & 0 \\ 0 & 0 & -y & 0 & -z & -y \\ 0 & 0 & 0 & z & 0 & -x \\ 0 & 0 & 0 & y & x & 0 \end{pmatrix}, \tag{6a}$$

$$a(-1,0) = -a^T(1,0), \tag{6b}$$

$$a(0,-1) = -a^T(0,1), \tag{6c}$$

$$a(-1,-1) = -a^T(1,1), \tag{6d}$$

$$a(1,0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & -z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6e}$$

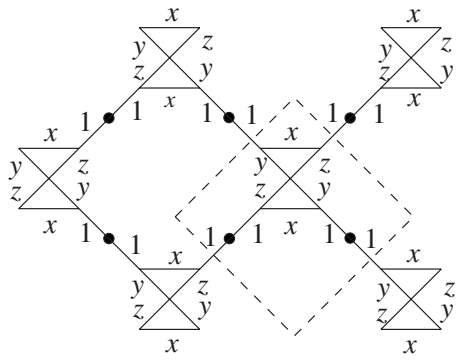

FIG. 3. An extended kagomé lattice $\mathcal{L}'$ constructed by inserting a decorating site attached to two inserted edges of weight 1 as shown. The decorating sites are denoted by solid circles. The unit cell is the region bounded by dashed lines. Repeating unit cells form a square lattice.

$$a(0,1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6f}$$

$$a(1,1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6g}$$

where the superscript T denotes the matrix transpose.

The evaluation of the determinant in Eq. (4) gives the surprisingly simple result

$$\det[M(\theta,\phi)] = 16x^2y^2z^2. \tag{7}$$

The substitution of Eq. (7) into Eq. (4) now yields Eq. (1). The expression (7) and result (1) have previously been obtained for $x=y=z=1$ in [6] for pure dimer enumerations.

III. VERTEX-MODEL APPROACH

The kagomé dimer problem can also be solved using a vertex model approach which is conceptually simpler. This involves the mapping of the dimer problem onto a vertex model for which the solution is known.

The first step of the mapping is to introduce the extended kagomé lattice $\mathcal{L}'$ of Fig. 3. The extended lattice $\mathcal{L}'$ is constructed from $\mathcal{L}$ by introducing $4N/3$ extra lattice edges with weight 1 and $2N/3$ new (decorating) sites as shown. By inspection it is clear that a bijection exists between dimer configurations on $\mathcal{L}$ and $\mathcal{L}'$. This permits us to consider instead the dimer problem on $\mathcal{L}'$.

The dimer problem on $\mathcal{L}'$ is next mapped onto a vertex model.

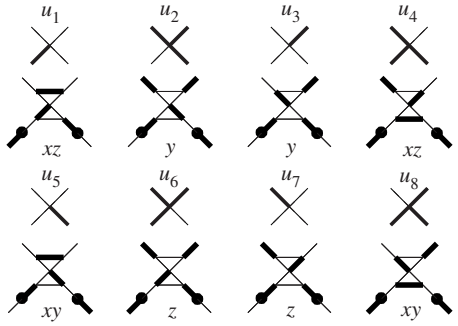

FIG. 4. Mapping between vertex and dimer configurations and the corresponding weights.

The extended lattice $\mathcal{L}'$ consists of $N/3$ unit cells each of which is the region bounded by dashed lines shown in Fig. 3. The unit cells form a square lattice $\mathcal{S}$. We next map dimer configurations on $\mathcal{L}'$ onto vertex configurations on $\mathcal{S}$, by regarding the four edges extending from a unit cell of $\mathcal{L}'$ as the four edges incident to a site on $\mathcal{S}$. To each extending edge on $\mathcal{L}'$ covered by a dimer, draw a *bond* on the corresponding edge on $\mathcal{S}$, and to each extending edge not covered by a dimer, leave the corresponding edge *empty*. Then, as shown in Fig. 4, dimer coverings of a unit cell are mapped onto vertex configurations on $\mathcal{S}$. Since the number of bonds extending from each vertex is either 1 or 3, which is an odd number, we are led to the *odd eight-vertex model* considered in [18].

Vertex weights of the odd eight-vertex model can be read off from Fig. 4 as

$$u_1 = xz, \quad u_2 = y, \quad u_3 = y, \quad u_4 = xz,$$

$$u_5 = xy, \quad u_6 = z, \quad u_7 = z, \quad u_8 = xy. \tag{8}$$

The per-vertex eight-vertex model free energy is then

$$f_{8v}(x,y,z) = \lim_{N\to\infty} \frac{1}{N/3} \ln Z_{\text{kagomé}}(x,y,z). \tag{9}$$

Comparing Eq. (9) with Eq. (3), we obtain the equivalence

$$f_{\text{kagomé}}(x,y,z) = \left(\frac{2}{3}\right) f_{8v}(x,y,z). \tag{10}$$

Now the weights (8) satisfy the free-fermion condition

$$u_1u_2 + u_3u_4 = u_5u_6 + u_7u_8 \tag{11}$$

for which the per-vertex eight-vertex model free energy has been evaluated in [18] as

$$f_{8v} = \frac{1}{16\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln F(\theta,\phi), \tag{12}$$

where

$$F(\theta,\phi) = 2A + 2D\cos(\theta-\phi) + 2E\cos(\theta+\phi) + 4\Delta_1 \sin^2\phi + 4\Delta_2 \sin^2\theta,$$

with

$$A = (u_1u_3 + u_2u_4)^2 + (u_5u_7 + u_6u_8)^2,$$

$$D = (u_5u_7)^2 + (u_6u_8)^2 - 2u_1u_2u_3u_4,$$

$$E = -(u_1u_3)^2 - (u_2u_4)^2 + 2u_5u_6u_7u_8,$$

$$\Delta_1 = (u_1u_2 - u_5u_6)^2,$$

$$\Delta_2 = (u_3u_4 - u_5u_6)^2. \tag{13}$$

The solution (1) is now obtained by substituting Eqs. (13) into Eqs. (12) and (10).

IV. DIMER-DIMER CORRELATION

The dimer-dimer correlation function can be evaluated by either considering a perturbation of the Pfaffian as in [19,20] or by applying the Grassmannian method of [21,22]; details of both approaches will be given elsewhere. Here we sketch steps in the Pfaffian computation.

Define for the lattice edge connecting sites i and j in unit cell at $\mathbf{r}=(r_x,r_y)$ an edge *vacancy* number

$$n_{ij,\mathbf{r}} = 1, \quad \text{if } ij \text{ is empty},$$
$$= 0, \quad \text{if } ij \text{ is occupied}, \tag{14}$$

where $\langle\cdot\rangle$ denotes the configurational average. Then, the correlation function between two dimers on edges ij in cell $\mathbf{r}_1$ and $k\ell$ in cell $\mathbf{r}_2$ is

$$c(ij,\mathbf{r}_1;k\ell,\mathbf{r}_2) = \langle \bar{n}_{ij,\mathbf{r}_1}\bar{n}_{k\ell,\mathbf{r}_1}\rangle - \langle \bar{n}_{ij;\mathbf{r}_1}\rangle\langle \bar{n}_{k\ell;\mathbf{r}_2}\rangle. \tag{15}$$

To make use of Eq. (15) we need to compute the dimer generating function with specific edge(s) missing. Let A be the antisymmetric matrix derived from the Kasteleyn orientation, and let A' denote the antisymmetric matrix derived from A with edge ij—say, in computing $\langle\bar{n}_{ij}\rangle$—missing. Write

$$Z = \mathrm{Pf}A,$$

$$Z' = \mathrm{Pf}A' = \mathrm{Pf}[A+\Delta], \tag{16}$$

where Δ is the matrix with zero elements everywhere except the ij element is $-A_{ij}$ and the ji element is $-A_{ji}(=A_{ij})$. Then

$$\langle\bar{n}_{ij}\rangle = Z'/Z = \mathrm{Pf}A'/\mathrm{Pf}A \tag{17}$$

and

$$\langle\bar{n}_{ij}\rangle^2 = \frac{\det A'}{\det A} = \frac{\det[A(I+G\Delta)]}{\det A} = \det(I+G\Delta), \tag{18}$$

where $G=A^{-1}$ is the Green's function matrix and I the identity matrix.

In computing Eq. (18) we need only to keep those row(s) and column(s) in Δ and A^{-1} where elements of Δ are nonzero. In addition, in the interior of a large lattice, the correlation depends only on the difference $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2=\{r_x,r_y\}$, so elements of G are given by

$$G(\mathbf{r}) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} d\theta\, d\phi e^{i(r_x\theta+r_y\phi)}A^{-1}(\theta,\phi). \tag{19}$$

These considerations lead to the explicit evaluations of Eq. (18) and, hence, the correlation (15).

Particularly, due to the fact that elements in $A^{-1}(\theta,\phi)$ contain only a monomial of $e^{\pm i\theta}$ and $e^{\pm i\phi}$, a consequence of the fact that the determinant $\det A$ is given by the simple expression (7), the integral (19) vanishes for $|r_{1x}-r_{2x}|>1$ or $|r_{1y}-r_{2y}|>1$. This leads to the result

$$c(ij,\mathbf{r}_1;k\ell,\mathbf{r}_2) = 0, \quad |\mathbf{r}_1-\mathbf{r}_2| \geqslant 2. \tag{20}$$

The absence of the dimer-dimer correlation beyond a certain distance, which is also found in the Sutherland-Rokhsar-Kivelson state of a quantum dimer model [13], is a property unique to the kagomé lattice. This underscores the special role played by the kagomé lattice in the statistical mechanics and quantum physics of lattice systems.

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